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Letter to the Editor

On the virtual acoustical source in mapped infinite element

L.X. Li[a,b,*], J.S. Sun[b], H. Sakamoto[b]

[a] School of Civil Engineering and Mechanics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, People's Republic of China

[b] Technical Research Institute, AOKI Corporation, Kaname 36-1, Tsukuba, Ibaraki 300-2622, Japan

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1. Introduction

The infinite element approach is effective in solving unbounded wave problems. It models the unbounded region in its entirety by using elements of infinite extent, and the non-radiation condition is therefore readily accommodated. In the past 20 years, a number of infinite element schemes have appeared, which can roughly fall into mapped element and unmapped element. In unmapped infinite element [1,2], the shape functions are directly constructed within physical element. Although the convergence and accuracy of this kind of elements are assured by interpolation functions, they often involve complex integration procedures in forming the system matrices because of the infiniteness of real elements. In mapped infinite element [3–10], a geometry mapping is first introduced, and the field variable is then interpolated within the parent element. In particular, the mapped wave envelope infinite element enables all the system matrices to be evaluated by using the standard Gauss quadrature. Another advantage of this envelope element is that the system matrices can be separated in terms of the power of frequency and therefore can be easily used to solve transient problems, although the symmetric nature is not preserved in them, which is the only drawback of this element.

Mapped infinite element has been well developed in the work by Cremers et al. [3], Astley et al. [4] and others [5–10]. Cremers et al. [3] presented a variable order infinite acoustic wave envelope element and did many numerical experiments of two dimensions and three dimensions to show the validity of their element. To demonstrate the effect of a source shift on numerical modelling of an amplitude decay, a one-dimensional example was investigated in the appendix. Indeed, the exact solution was finally achieved as the order of infinite element was increased up to seven, but in our view, this simple test just exposed some problems inherent in their elements.

*Corresponding author. School of Civil Engineering and Mechanics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, People's Republic of China.

E-mail address: luxianli@mail.xjtu.edu.cn (L.X. Li).

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2. Basic ideas in Cremers' element

2.1. Concept of virtual acoustical source

In Cremers et al.'s work [3] the virtual source played an important role in explaining the geometry mapping

$$x = \frac{2(x_{II} - x_I)}{1 - t} + (2x_I - x_{II}). \tag{1}$$

On letting

$$x_0 = 2x_I - x_{II}, \tag{2}$$

$$a = x_{II} - x_I = x_I - x_0, \tag{3}$$

$$r = x - x_0, \tag{4}$$

Cremers et al. [3] obtained the inverse mapping of Eq. (1) as

$$t = 1 - \frac{2a}{r}. \tag{5}$$

Astley et al. [4] called x_0 the location of virtual acoustical source whereas Cremers et al. [3] briefly referred to it as the location of source. r is therefore the distance to any arbitrary point in the element from the source at x_0. In this article, the notations have the same meaning as in Ref. [3] except those specially specified.

2.2. Shape functions in Cremers' element

Cremers et al. selected Lagrangian version of shape functions. The typical first and second order take the form

$$N_1 = \frac{1 - t}{2}, \quad \text{first order} \tag{6}$$

and

$$\begin{aligned} N_1 &= t(t - 1), \\ N_2 &= 1 - t^2, \end{aligned} \quad \text{second order.} \tag{7}$$

For instance, Eqs. (5) and (6) give

$$r(t = 0) = 2a, \tag{8a}$$

$$p(t = 0) = \tfrac{1}{2}p(t = -1), \tag{8b}$$

which means that the pattern of first order can only accurately describe the wave decay in which case the amplitude at $x = x_{II}$ is half that at $x = x_I$. Indeed, it is true of the case in which the real source coincides with the virtual source at $x = x_0$ because of

$$r_{II} = x_{II} - x_0 = 2(x_I - x_0) = 2r_I. \tag{9}$$

In other words, the virtual source cannot be located at the discretion of the analyst for the purpose that the first order element could give satisfactory results.

Similarly, for instance, Eqs. (5) and (7) give

$$r(t=-\tfrac{1}{2})=\tfrac{4}{3}a, \tag{10a}$$

$$p(t=-\tfrac{1}{2})=\tfrac{3}{8}p(t=-1)+\tfrac{3}{4}p(t=0). \tag{10b}$$

Due to the fact that Eq. (10a) is compatible with Eq. (10b) for $1/r$-type wave only on the condition that Eq. (8b) holds, they arrive at the same evaluation as Eqs. (8a) and (8b). Of course, the contradiction between Eqs. (10a) and (10b) will be mitigated through the increase of independent variables from 1 (single $p(t=-1)$) to 2 (dual $p(t=-1)$ and $p(t=0)$), for example.

Following the above comments, the reasonableness of the virtual source is so far doubtful as its presence brings about the contradiction between geometry mapping and shape functions within mapped infinite element for the more general cases. Probably, this is the reason why Cremers et al. [3] had to adopt up to seventh order infinite element to remedy the deviation caused by moving of the infinite element only a bit farther (x_0 varies from 0 to 1) whereas the exact solution can be obtained by using first element pattern conversely. As a matter of fact, through looking at the geometry of infinite element, one cannot find so great a difference between these two cases, and moreover, the farther the infinite element away from the real source (the origin in the example), the more accurate mapped infinite element should model the field from the knowledge of infinite element. Unfortunately, Table A1 of Cremers et al. [3] gave quite unexpected results against our common sense. In our opinion, this is owing to rather an improper form of shape functions restricted by the unnecessary concept of virtual source than an improper location of the virtual source believed by Cremers et al. [3].

3. A scheme to resolve this problem

If the concept of virtual source and the related notations such as x_0, a and r are discarded, Eq. (5) can be rewritten as

$$t=1-\frac{2(x_{\mathrm{II}}-x_{\mathrm{I}})}{x-(2x_{\mathrm{I}}-x_{\mathrm{II}})}. \tag{11}$$

Eqs. (1) and (11) can achieve a well-posed geometry mapping along an infinite direction from x_{I} to x_{II} for any values of x_{II} and x_{I} since the following relationships hold:

$$x(-1)=x_{\mathrm{I}}, \tag{12a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2(x_{\mathrm{II}}-x_{\mathrm{I}})}{(1-t)^2}, \tag{12b}$$

so that the monotonicity of x with t in the interval $[-1,1)$ is ensured. In practice, x_{II} and x_{I} have the same sign and thereby $|x_{\mathrm{II}}|>|x_{\mathrm{I}}|$ as the origin is not the point of interest in mapped infinite element.

As previously mentioned, Eqs. (6) and (7) are not the proper versions for the general value of x_{II}. Based on the fact that the shape function of first order element is non-unique subject to

$$N_1(-1) = 1, \quad N_1(1) = 0 \tag{13}$$

and noting that

$$\frac{x_{\mathrm{I}}}{x} = \frac{1-t}{-2t+\alpha(1+t)}, \tag{14}$$

a more proper form of first order shape function may be

$$N_1(t) = \frac{1-t}{-2t+\alpha(1+t)}, \quad \text{first order} \tag{15}$$

as proposed by the authors in Ref. [11] where $\alpha = x_{\mathrm{II}}/x_{\mathrm{I}}$. This kind of shape function incorporates the effect of the location of x_{II} and therefore represents the essence of mapped infinite element.

Apparently,

$$p(t) = N_1(t)p(t=-1) \Leftrightarrow p(x) = \frac{x_{\mathrm{I}}}{x}p(x_{\mathrm{I}}) \tag{16}$$

arrives at the exact solution

$$p(x) = 1/x \tag{17}$$

given the prescribed boundary condition as (A2) at $x = x_{\mathrm{I}}$ in Ref. [3] wherever x_{II} locates in the infinite element.

For second order element, following the criterion proposed in Ref. [11] yields

$$N_1(t) = -tS_1, \quad N_2(t) = (1+t)S_2, \quad \text{second order,} \tag{18}$$

where S_1 and S_2 were termed shape factors by the authors and take the form

$$S_1 = \frac{1-t}{2}\left(\frac{2}{-2t+\alpha(1+t)}\right)^2, \quad S_1 = (1-t)\left(\frac{\alpha}{-2t+\alpha(1+t)}\right)^2 \tag{19}$$

in terms of local co-ordinate t. Eq. (18) assures virtually the dipole expansion for the second order element for any value of α.

4. Numerical results and discussion

Cremers et al. [3, Table A1] listed the results when $a = 1$ and $x_0 = 1$ ($x_{\mathrm{I}} = 2$ and $x_{\mathrm{II}} = 3$), in which an order one higher than the interpolation order was used in the Gaussian quadrature. For the purpose of comparison, the authors programmed the Cremers' theory. The codes have proved to obtain the same results as listed in Table A1 for the identical case. If a keeps the value of 1 and the virtual source is moved much farther away from the real source, it is predictable that higher

Table 1
Numerical results when $a = 1$ and $x_0 = 8$ ($x_{\mathrm{I}} = 9, x_{\mathrm{II}} = 10$)

x	Exact	Present		Cremers et al. [3]					
		1st	2nd	1st	2nd	7th	8th	9th	10th
9	0.11111	0.11115	0.11115	0.03518	0.07484	0.11102	0.11109	0.11112	0.11114
10	0.10000	0.10003	0.10006	0.01759	0.05920	0.09998	0.10003	0.09999	0.10003
100	0.01000	0.01000	0.01003	0.00038	0.00175	0.00873	0.00922	0.00956	0.01021
500	0.00200	0.00200	0.00201	0.00007	0.00033	0.00170	0.00181	0.00188	0.00204
1000	0.00100	0.00100	0.00100	0.00004	0.00016	0.00085	0.00090	0.00094	0.00102

than seventh order element is needed to achieve the exact solution for Cremers' element. This prediction is immediately verified by the numerical results listed in Table 1. Using up to tenth order element, Cremers' element gave results with 2% error at $x = 1000$ for the case of $a = 1$ and $x_0 = 8$. Here, 10-point Gaussian quadrature is used in the procedure of numerical integration. Due to the instability in, and the need of higher than tenth order mapped infinite element (also in need of much higher order Gaussian quadrature) to get the satisfactory results, the disadvantage of Cremers' element shows up strongly in this simple example.

Another fact showing the drawback of Cremers' element is that the value of α is within certain bounds but arbitrarily selected. This can be well accounted for by re-analyzing second order mapped infinite element.

Assuming that $p(x_{\mathrm{I}})$ and $p(x_{\mathrm{II}})$have been obtained as their analytical solution $1/x_{\mathrm{I}}$ and $1/x_{\mathrm{II}}$, based on Eq. (7), the interpolated field variable will be

$$p(t) = \frac{(1-t)[(2-\alpha)t+2]}{2\alpha x_{\mathrm{I}}}. \tag{20}$$

As is known, $p(x)$ should monotonically decrease as t varies within the interval $[-1,1)$. That is to say, it is always required to preserve

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{2(2-\alpha)t-\alpha}{2\alpha x_{\mathrm{I}}} \leqslant 0 \tag{21}$$

for any value of α greater than 1. Unfortunately, for the case in Table 1 where $\alpha = 10/9$ we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{10x_{\mathrm{I}}} > 0 \tag{22}$$

at $t = 3/4$ ($x = 16x_{\mathrm{I}}/9$). This is against the decay of wave in infinite element. Careful study indicates that the choice of α follows:

$$\tfrac{4}{3} \leqslant \alpha \leqslant 3, \tag{23}$$

where the upper bound is obtained using the required dipole expansion. It is hard to obtain such an equation as Eq. (23) for higher order Cremers' element. However, this kind of effect can be clearly seen from the comparison between Table 2 ($\alpha = \frac{13}{9} \approx 1.44$) and Table 1 ($\alpha = \frac{10}{9} \approx 1.11$).

Finally, numerical test is performed for the case of ($\alpha = \frac{23}{9} \approx 2.56$) to show the feasibility of selecting a value of α greater than 2 on the condition that $|x_0|$ keeps the value of 5. Evidently, the results listed in Table 3 are better than those in Table 2.

Table 2
Numerical results when $x_I = 9$ and $x_{II} = 13$ $(a = 4, x_0 = 5)$

x	Exact	Present		Cremers et al. [3]					
		1st	2nd	1st	2nd	3rd	4th	5th	6th
9	0.11111	0.11111	0.11111	0.09471	0.11019	0.11106	0.11111	0.11111	0.11111
13	0.07692	0.07692	0.07692	0.04735	0.07372	0.07709	0.07698	0.07692	0.07692
100	0.01000	0.01000	0.01000	0.00399	0.00764	0.00919	0.00975	0.00993	0.00998
500	0.00200	0.00200	0.00200	0.00077	0.00149	0.00181	0.00194	0.00198	0.00199
1000	0.00100	0.00100	0.00100	0.00038	0.00074	0.00090	0.00097	0.00099	0.00100

Table 3
Numerical results when $x_I = 9$ and $x_{II} = 23$ $(a = 14, x_0 = -5)$

x	Exact	Present		Cremers et al. [3]					
		1st	2nd	1st	2nd	3rd	4th	5th	6th
9	0.11111	0.11111	0.11111	0.10749	0.11105	0.11111	0.11111	0.11111	0.11111
23	0.04348	0.04348	0.04348	0.05374	0.04249	0.04344	0.04349	0.04348	0.04348
100	0.01000	0.01000	0.01000	0.01433	0.00878	0.01022	0.00997	0.01000	0.01000
500	0.00200	0.00200	0.00200	0.00298	0.00167	0.00208	0.00198	0.00200	0.00200
1000	0.00100	0.00100	0.00100	0.00150	0.00083	0.00104	0.00099	0.00100	0.00100

To sum up, the accuracy of Cremers' element is strongly dependent on the difference of distance between virtual acoustical source and real source, and, meanwhile, more than fourth order mapped infinite element is often needed to model even a quite simple example. Besides, close attention must be paid to the choice of α value to prevent uncertainties as in Table 1. To our excitement, the improvement suggested by the authors displays good performance in many respects. This idea has been successfully applied to solve the two-dimensional problems [11] and can be extended to three dimensions.

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